Quantum Mechanics is (Symplectic) Classical Mechanics

Nicholas Andrzejkiewicz

SUNY Polytechnic

Some Linear Algebra

In the wavefunction/state vector formulation of quantum mechanics, wavefunctions between two times (0 and t) are related by a linear Unitary operator:

$$A(t)\psi(0) = \psi(t), A(t) \in U(V)$$

Where $U(V)$ is the set of unitary operators acting on the Hilbert space V. Meanwhile in classical mechanics, states in linear phase spaces evolve by linear symplectomorphisms, in the lie group $Sp(2n, V)$. This time V is a real vector space of dimension 2n, and the operators are required to preserve a symplectic form rather than a Hermitian form. In finite (n) dimensions, the Hermitian form H's action can be expressed as the matrix product:

$$(\psi, \phi) = \psi^T Q\overline{\phi} = \psi^T\overline{\phi}, Q = I$$

$$\psi = \begin{bmatrix} x_1 + ip_1 \\ \vdots \\ x_n + ip_n \end{bmatrix}, \phi = \begin{bmatrix} x_1' + ip_1' \\ \vdots \\ x_n' + ip_n' \end{bmatrix}$$

$$(\psi, \phi) = \begin{bmatrix} x_1 + ip_1 \cdots x_n + ip_n \end{bmatrix} \begin{bmatrix} x_1' - ip_1' \\ \vdots \\ x_n - ip_n' \end{bmatrix}$$

$$= (x_1x_1' + p_1p_1' + \cdots + x_nx_n' + p_np_n') + i(p_1x_1' - x_1p_1' + \cdots + p_nx_n' - x_np_n')$$

However when written as a 2n dimensional real vector space, the real part of the hermitian form resembles the dot product while the imaginary part is the symplectic product in question:

$$\psi_{\mathbb{R}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ p_1 \\ \vdots \\ p_n \end{bmatrix}, \phi_{\mathbb{R}} = \begin{bmatrix} x_1' \\ \vdots \\ x_n' \\ p_1' \\ \vdots \\ p_n' \end{bmatrix}$$

$$\psi_{\mathbb{R}}^T\phi_{\mathbb{R}} = (x_1x_1' + p_1p_1' + \cdots + x_nx_n' + p_np_n')$$

$$\omega(\psi_{\mathbb{R}}, \phi_{\mathbb{R}}) = \psi_{\mathbb{R}}^T \begin{bmatrix} 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & -1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \phi_{\mathbb{R}}$$

$$= \psi_{\mathbb{R}}^T \begin{bmatrix} \mathbf{0}_n & -\mathbf{I}_n \\ \mathbf{I}_n & \mathbf{0}_n \end{bmatrix} \phi_{\mathbb{R}} = \psi_{\mathbb{R}}^T \mathbf{J}_{2n} \phi_{\mathbb{R}}$$

Where $\mathbf{0}_n$ is the $n \times n$ matrix of zeroes, and $\mathbf{I}_n$ is the $n \times n$ identity. The matrix $\mathbf{J}_{2n}$ only makes sense in even dimensions, which ties it to linear algebra with complex numbers - it is called the symplectic form interchangeably with the binary function ω. Immediately it follows that the unitary group is an intersection:

$$U(n) = Sp(2n) \cap O(2n)$$

Known as the two-out of three property (Sepanski, *Compact Lie Groups*).

Symplectic Geometry and Classical Mechanics

Orthogonal matrices, those that preserve the dot product are also known as rotations. While rotations preserve the length $\psi_{\mathbb{R}}^T\psi_{\mathbb{R}} = (O\psi_{\mathbb{R}})^T(O\psi_{\mathbb{R}})$ for any $\psi_{\mathbb{R}} \in \mathbb{R}^{2n}$, there is no analogue of length for the Symplectic product $\omega(-,-)$ since it is antisymmetric ($\omega(\psi_{\mathbb{R}}, \psi_{\mathbb{R}}) = 0$). The value of the Symplectic product shown previously can be seen as a sum of 2×2 determinants. It is also important that $\omega(-,-)$ "knows" about the division into the x and p coordinates by swapping them:

$$\omega(\psi_{\mathbb{R}}, -) = \begin{bmatrix} p_1 \cdots p_n & -x_1 \cdots -x_n \end{bmatrix}$$

The geometrical meaning becomes clear when considering real-valued functions on $\mathbb{R}^{2n}$, interpreting their values as energy associated to the state $\psi_{\mathbb{R}}$ - positions and momenta at an instant in time. E.g. on $\mathbb{R}^2$:

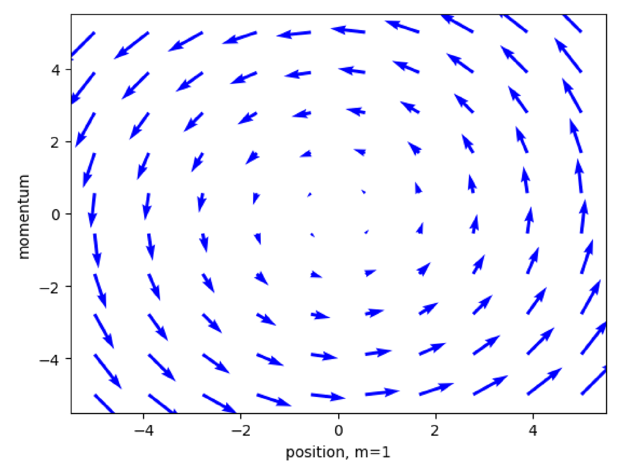
$$H(x,p) = \frac{1}{2}x^2 + \frac{p^2}{2m}, dH = xdx + \frac{p}{m}dp$$

Interpreting the differential as a row vector:

$$\begin{bmatrix} x & \frac{p}{m} \end{bmatrix} = \omega(\tfrac{d\psi_{\mathbb{R}}}{dt}, -)$$

Defining the column vector solution $\frac{d\psi_{\mathbb{R}}}{dt}$ to this equation to be the time derivative of the state, treated as a vector field:

$$\psi_{\mathbb{R}} = \begin{bmatrix} x \\ p \end{bmatrix} \mapsto \frac{d\psi_{\mathbb{R}}}{dt} = \begin{bmatrix} -\frac{p}{m} \\ x \end{bmatrix}$$



This more clearly shows the function H to be the energy of an oscillating mass on a spring with mass m. The Symplectic form plays the role of converting energy functions into time evolution vector fields, and their time evolution can be interpreted as preserving volumes in position-momentum phase space. Finally, vector fields can be interpreted as differential operators:

$$\begin{bmatrix} -\frac{p}{m} \\ x \end{bmatrix} \to -\frac{p}{m}\frac{\partial}{\partial x} + x\frac{\partial}{\partial p} = D$$

Permitting time-evolving not just points, but functions too:

$$e^{tD}f(x,p,0) = f(x,p,t)$$

or in one dimension, Taylor expanding:

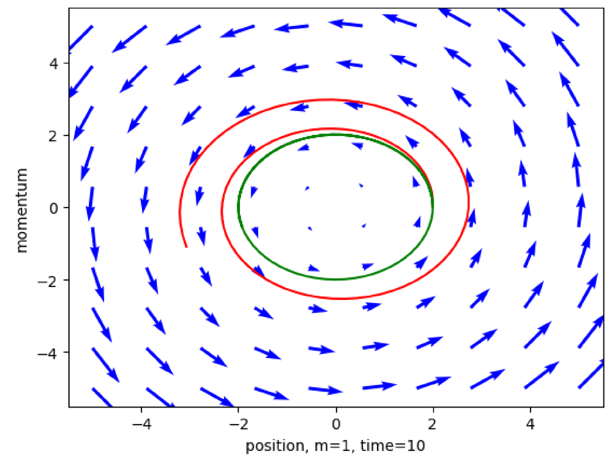
$$e^{-c\frac{d}{dx}}f(x) = f(x-c)$$

Symplectic Numerical Integration

Given a starting point (x_0, y_0) and a slope field $\frac{dy}{dx} = f(x,y)$, the function $y(x)$ satisfying $y(x_0) = y_0$ following the slopes is simply approximated with the Euler step method:

$$y_1 = y_0 + hf(x_0, y_0), \cdots, y_{n+1} = y_n + hf(x_n, y_n)$$

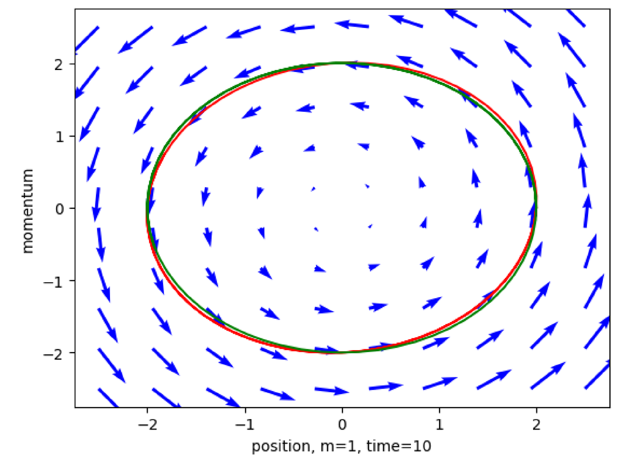
With uniform x-step sizes h, $x_n = x_0 + nh$ and y_n is the approximate height of the graph at x_n. Applied to the oscillator:



$$x_{n+1} = x_n - h\frac{\partial H}{\partial p}(x_n, p_n)$$
$$p_{n+1} = p_n + h\frac{\partial H}{\partial p}(x_n, p_n)$$

The green trajectory is the exact solution, red is the curve constructed from Euler steps with step size 0.1. Evidently Euler's method does not respect the necessary conservation of energy. A different method designed to preserve the Symplectic form is needed:

$$p_{n+1} = p_n - h\frac{\partial H}{\partial x}(x_n, p_n)$$
$$x_{n+1} = x_n + h\frac{p_{n+1}}{m} = x_n + h\frac{p_n}{m} - h^2\frac{\partial H}{\partial x}(x_n, p_n)$$



zoomed in to show accuracy

This time, the new momentum must be computed first since the new position depends on it. Write a matrix representing the step:

$$\begin{bmatrix} x_{n+1} \\ p_{n+1} \end{bmatrix} = \begin{bmatrix} 1 - h^2/m & h/m \\ -h & 1 \end{bmatrix} \begin{bmatrix} x_n \\ p_n \end{bmatrix}$$

Since in two dimensions, the Symplectic condition is preserving a single determinant/area, whether this matrix M is Symplectic can be checked more simply if its determinant is 1, rather than plugging into $\omega(M(-), M(-)) = \omega(-,-)$:

$$M^T\mathbf{J}_2 M = \mathbf{J}_2$$

Deutsch's Algorithm

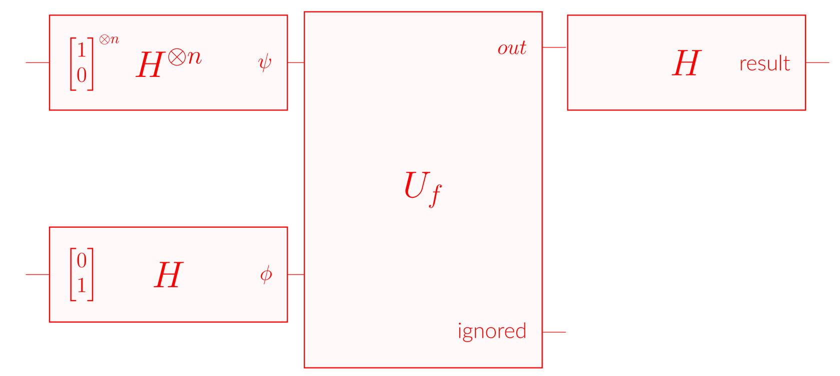
Since unitary matrices are a subset of symplectic matrices, the unitary "gate" operations of a quantum computer may be differentiated by a time parameter to give an analogue of the energy (Hamiltonian) vector field. 45-degree rotation is a unitary matrix:

$$H = \begin{bmatrix} cos(t\frac{\pi}{4}) & -sin(t\frac{\pi}{4}) \\ sin(t\frac{\pi}{4}) & cos(t\frac{\pi}{4}) \end{bmatrix}, t = 1$$

Which will be referred to as H from now. Also known in quantum computing as the Hadamard gate, we can differentiate it at 0 to obtain the matrix of a Hamiltonian vector field over $\mathbb{C}^2$:

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \mapsto \begin{bmatrix} 0 & -\frac{\pi}{4} \\ \frac{\pi}{4} & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -\frac{\pi}{4}c_2 \\ \frac{\pi}{4}c_1 \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Symplectic integration from time 0 to 1 recovers the Hadamard operator. Deutsch's Algorithm is represented by the following schematic:



Some notation: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\otimes n}$ represents a 2^n-tuple with only one nonzero component, similarly taking $H^{\otimes n}$ represents a $2^n \times 2^n$-block matrix whose components are products of components of copies of the matrix H. Basis-independently, these are thought of as tensors, but the block representation makes the unitarity of the operators involved manifest. The operator U_f depends on a boolean function $f(x_1, \cdots, x_n) \to \{0, 1\}$ of n variables. Each element of the tensor basis is some list of products of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, U_f sends such a product Ψ to $-1^{f(b(\Psi))}\Psi$. Extending it to all linear combinations of basis products defines the operator on the total space $\mathbb{C}^{2^n+1}$. The input $b(\Psi)$ is a list of 1's and 0's corresponding to the number and ordering of each basis vector in the constituent product. The ultimate output of this algorithm is a probability distribution that helps to decide if the function f is "balanced," outputting 0 for equally many inputs as 1.

To interpret the circuit as a Symplectic-mechanical system, we can find the Hamiltonian vector field/derivative matrix corresponding to each unitary operator. Time can be thought of as passing to the right, where after each application of a unitary operator, one unit of time has passed, and a new vector field takes over.

This way, we can verify properties of classical logic circuits with continuous time mechanical systems.

Future Directions

In all of the cases here, linear Symplectic mappings were considered - however more general area preserving mappings exist on other geometries, where both the geometry and the maps are nonlinear. In quantum mechanics and computing, only the unit sphere S^3 matters as the operators are linear, how the unit sphere maps to itself (length is preserved under unitaries so it must) determines the map for all of $\mathbb{C}^2$. It can be reduced even further, to the two-dimensional sphere S^2 by also considering linearity under multiplication by unit complex numbers, this reduction itself has an interesting geometry called the Hopf Fibration.

$$S^1 \hookrightarrow S^3 \to S^2$$

The phase spaces arising in ordinary mechanics are quite different, they extend to infinity due to the velocity tangent spaces of position configuration manifolds being linear - for the pendulum where positions are on a circle, this Symplectic manifold is an infinite cylinder. Any manifold whose tangent spaces at each point have a Symplectic form are such manifolds, but S^2 is an example where it itself is not the tangent space of another manifold - it is closed and finite. This result specializes to manifolds whose tangent spaces are complex vector spaces, which is the case for S^2, as well as manifolds associated to parameterized families of probability distributions.

There is more geometry in quantum mechanics, S^2 was an example of the projective space of "true" states in $\mathbb{C}^2$, but we can consider it for any $\mathbb{C}^{n+1}$ as $\mathbb{CP}^n$. Projectively, the tensor product between two different complex vector spaces spaces becomes:

$$\mathbb{CP}^n \times \mathbb{CP}^m \to \mathbb{CP}^{nm+n+m}$$

This is known to algebraic geometers as the Segre embedding, it comes with a great deal of hidden combinatorial structures coming from its polynomial construction.

Symplectic evolution appears in such other objects of physics as:

- the Electromagnetic Field
- Gauge Fields
- Fluids
- Plasmas
- Quantum Fields

The power of the Symplectic method comes from its utilization of symmetry, indeed all Symplectic manifolds with parameterized families of transformations comes with a "momentum mapping" which in the case of $\mathbb{C}^n$ maps the states to probability distributions. This will be a powerful tool for comparing linear problems to nonlinear ones, and understanding the nature of computation geometrically, and physical systems.

andrzen@sunypoly.edu