Sparks of 460

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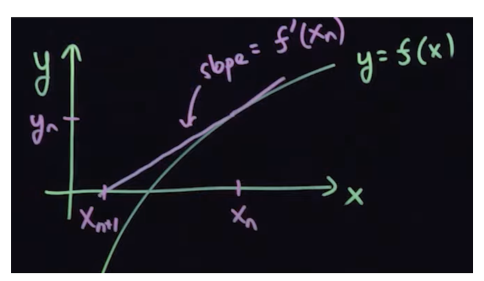
MAT 460 Numerical Differential Equations – Spring 2022, Department of Mathematics & Physics

Motivation

It has been my experience thus far in my math journey, that we typically use analytical methods and solutions to solve problems. In this class we learned many numerical methods to solve problems. Thus it became my motivation, to show some of them on simple to complex problems commonly used in other classes to demonstrate numerical methods many applications as an equivalent tool for solving math problems.

Newton $x_{n+1} = x_n - f(x_n)/f'(x_n)$

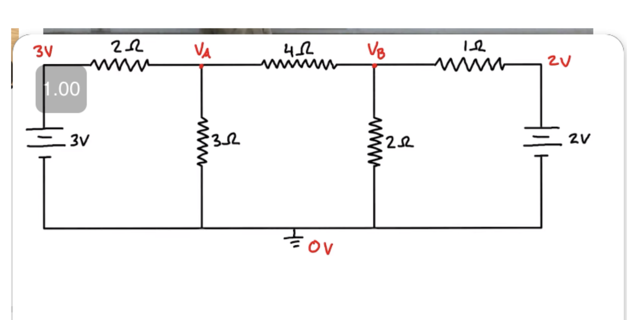
Newton's method solves for an x_{n+1} such that $f(x_{n+1}) \approx 0$ or rather $f(x_{n+1}) < tol$.



This can be used to solve for $\sqrt{2}$,using $f(x) = x^2 - 2$.
$x_{n+1} = x_n - f(x_n)/f'(x_n)$

for: $f(x) = x^2 - 2; f'(x) = 2x$; try: $x_n = 1/5 = 3/2.$ =>
$x_{n+1} = 3/2 - f(3/2)/f'(3/2) = 17/12; f(17/12) = X;$
try: $x_n = 17/12$ =>
$x_{n+1} = 17/12 - f(17/12)/f'(17/12) = 17/12$; f(17/12) = X;

Or it can be used in equilibrium problems such as solving for node voltages in a circuit.



$$f_1 = \frac{3-e_1}{2} - \frac{e_1}{3} - \frac{e_1-e_2}{4} = 0 \qquad f_2 = \frac{e_1-e_2}{4} - \frac{2-e_2}{1} - \frac{e_2}{2} = 0$$

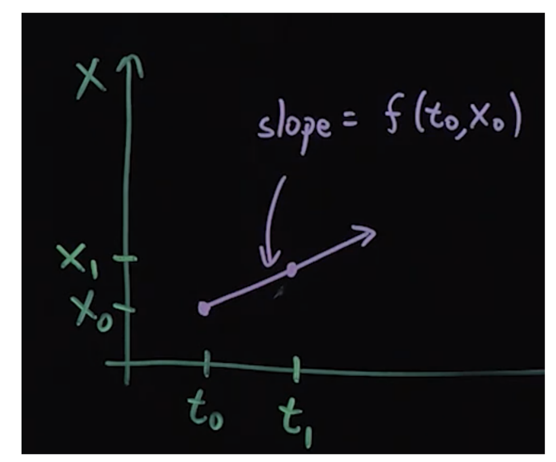
$$\vec{x_{n+1}} = \vec{x_n} - [f'(\vec{x_n})]^{-1} f(\vec{x_n}); \qquad \text{try: } \vec{x_0} = (e_1, e_2) = (2,1)$$

$$x_1 = (2,1) - \begin{bmatrix} \frac{-13}{12} & \frac{1}{4} \\ \frac{1}{4} & \frac{-7}{4} \end{bmatrix}^{-1} * f(\vec{(2,1)}) = (\tfrac{75}{44}, \tfrac{61}{44}) \approx (1.7045, 1.386)$$

The actual $\vec{x} = (e_1, e_2) = (1.7, 1.39)$.
This is only a primitive method and there are several more advanced methods, such as: Secant Method and Regula Falsi.

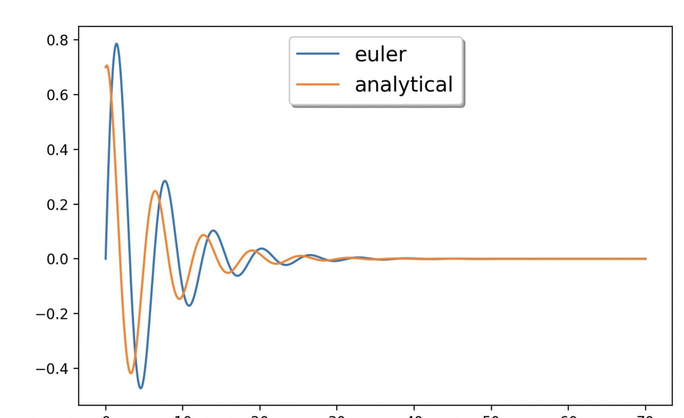
Euler $y_{n+1} = y_n + h * f(y_n)$

Euler's method is used for solving initial value differential equations.
When given an IVP such as $y' = y, y(0) = 1$ normally we solve for an analytical solution, $y(t) = e^t$.
With numerical methods we are solving for a set of $y(t) points = [y_0, y_1, ..., y_n]$.
We can derive an equation for the points, by linearizing the step from y_0 to y_1.



Thus we have: $y_1 = y_0 + h * y_0' = y_0 + h * f(y_0)$
generally we have: $y_{n+1} = y_n + h * f(y_n)$, this is Euler's method!
In particular for y' = y, we have: $y_{n+1} = y_n + h * y_n$

ex: $36y'' + 12y' + 37y = 0; y(0) = 0.7; y'(0) = 0.1$
Analytical: $y(t) = 0.7e^{\frac{-t}{6}} cos(t) + \frac{1.3}{6} e^{\frac{-t}{6}} sin(t);$
Euler:
$y_{n+1} = y_n + hy_n'; \quad y_{n+1}' = y_n' + hy_n''; \quad y_n'' = \frac{-37y_n - 12y_n'}{36};$

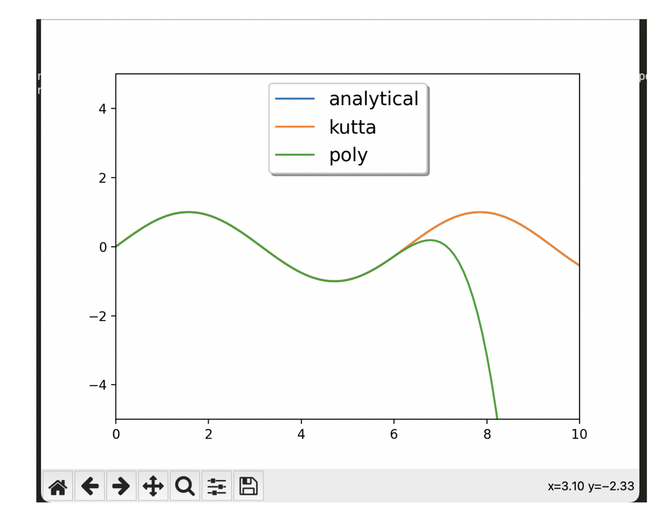


ex: $m_1 x_1'' = -kx_1 - K(x_1 - x_2)$
$m_2 x_2'' = -kx_2 - K(x_2 - x_1)$
Analytical: $\vec{x(t)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} [Acos(w_1 t) + Bsin(w_1 t)] + \begin{pmatrix} 1 \\ -1 \end{pmatrix} [Ccos(w_2 t) + Dsin(w_2 t)]$
Euler:
$x_{n+1} = x_n + hx_n'; \quad x_{n+1}' = x_n' + hx_n'';$
$$\vec{x_n''} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}^{-1} \begin{pmatrix} -(k+K) & K \\ K & -(k+K) \end{pmatrix} \vec{x_n}$$

Like the other methods, this is only the most basic form. We have more advanced forms such as Runge Kutta and Newmark Method.

Interpolation

Give a set of data points $[(x_0, y_0), ..., (x_m, y_m)]$ we can map it to a polynomial of size n. One popular method is Vandermonde.
$$\begin{pmatrix} 1 & x_0 & x_0^2 & ... & x_0^n \\ 1 & x_1 & ... & ... & ... \\ 1 & x_m & x_m^2 & ... & x_m^n \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ f_m \end{pmatrix}$$
Given a differential equation: $mx'' = -kx; m = k = 1; (x_0, v_0) = (0, 1);$ $y = sin(t)$
We sample points we generated from our Euler method, and you can see we get this polynomial: $p(x) = -0.366x^3 + 0.2311x^2 + 0.8973x + 0.009081$ which matches the Taylor series of a sine wave! $sin(x) = \Sigma \frac{(-1)^i x^{2i+1}}{(2i+1)!}$ Here is the picture of the polynomial:



Conclusion

There are many numerical methods, and they are an equivalent form of solve math problems. In a world of data, I can only see these tools being more relevant and useful.
I hope these simple examples, give people an easier understanding or refresher to the applicability and vailidity of these methods.

Acknowledgements & References

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