Mathematical Modeling of Spacex's Falcon 9 Booster Hoverslam Landing

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Introduction

This project will examine various numerical methods to model the Hoverslam maneuver performed during the landing of a Falcon 9 first stage rocket booster. The 2nd order nonlinear ordinary delay differential equation with constraints consists of gravity, air resistance and thrust terms and will be modeled using Runge-Kutta, Euler Implicit, and Euler Symplectic numerical methods. The numerical methods employed will be tested with varying step size in Python and presented to show that they are suitable methods to model this special type of differential equation application. They will be used to find the optimal ignition time of the booster, in order to land with a vertical velocity ($\dot{h}$) and height (h) of about 0.

Differential Equation

The second order delay differential equation with constraints to model the vertical acceleration of the booster is given as

$$\begin{aligned}\ddot{h} &= \frac{-GM_{\oplus}}{(R_{\oplus}+h)^2} + \frac{1}{2m(t)}\rho C_D A\dot{h}^2 - \frac{T}{m(t)}\\ \ddot{h} &= Gravity(h) + AirResistance(t,h,\dot{h}) + Thrust(t)\\ m(t) &= m_d + m_p - b(t_{burning})\end{aligned} \tag{1}$$

where G, $M_{\oplus}$, and $R_{\oplus}$ are astronomical coefficients associated with the Earth. For the Air Resistance term, $\rho(h)$ represents the density of air as a function of vertical height, and C_D is the drag coefficient of air. The Falcon 9 booster has a circular cross-sectional area with a radius of 1.83m and it's mass varies with the function m(t) shown above. Finally, the Thrust term consists of T which is the thrust produced by the booster in kN. The differential equation is described as a delay differential equation because the final term (Thrust) is only added to the equation for $t \geq t_{ignite}$. The initial conditions of $\dot{h}$ and h are known and serve as the starting point for the numerical methods. The integration is to be halted if the ground is reached or if the propellant is exhausted (m_p). These boundaries will vary as the time of ignition is changed to find the optimal ignition time.

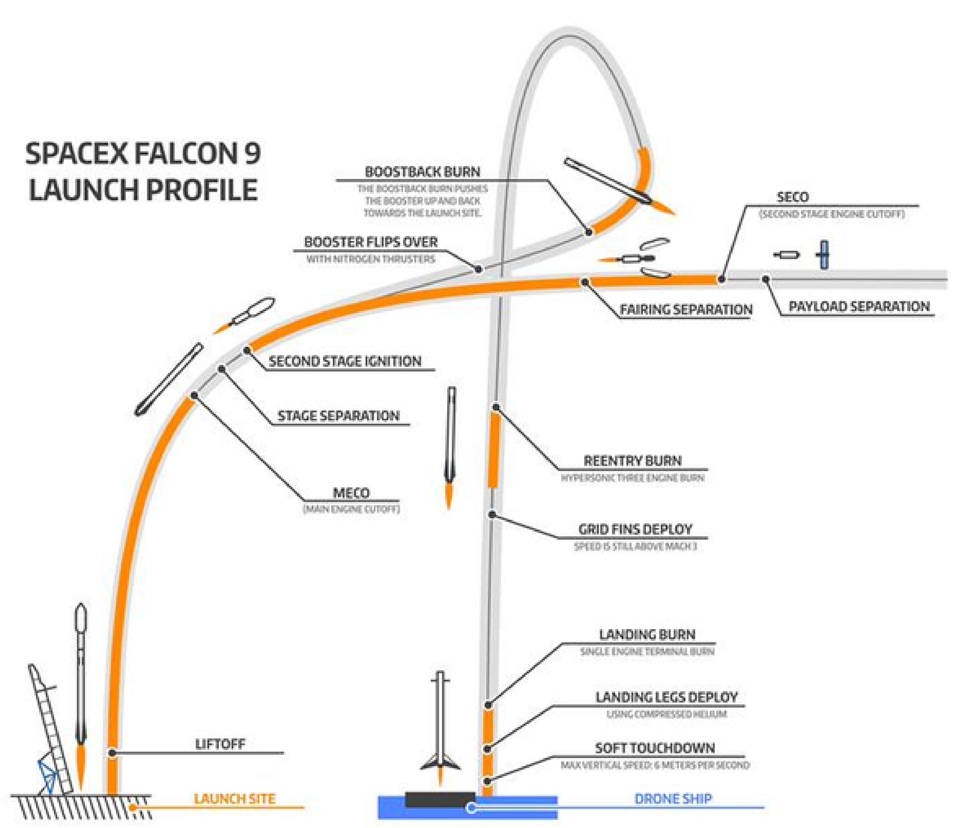

Fig. 1: Falcon 9 Booster Flight Path

Numerical Methods

The 2nd order differential equation that is used in this situation and throughout mechanics is of the form:

$$\ddot{h} = f(t, h, \dot{h}), \tag{2}$$

where, the motion is directed inwards towards the center of the Earth.

1 Euler Method

The Euler Method, which he published in 1768, is used to evaluate a first-order differential equation. It can be transformed to be applicable to 2nd order differential equations, and the explicit Euler scheme is shown in equation 3.

$$\begin{aligned}h_{n+1} &= h_n + \dot{h}_0\Delta t,\\ \dot{h}_{n+1} &= \dot{h}_n + f(t_n, h_n, \dot{h}_n)\Delta t\end{aligned} \tag{3}$$

1.1 Symplectic Euler Scheme

An explicit scheme finds values at a later time based on the values found at previous times. This method can be improved by combining both explicit and implicit schemes, creating a smyplectic scheme. The Euler symplectic scheme is shown below in equation 4.

$$\begin{aligned}\dot{h}_{n+1} &= \dot{h}_n + f(t_n, h_n, \dot{h}_n)\Delta t\\ h_{n+1} &= h_n + \dot{h}_{n+1}\Delta t\end{aligned} \tag{4}$$

1.2 Implicit Euler Method

The Euler method can also be manipulated into an implicit scheme that calculates the system at a future time from the given system at present and future times. In order to make this method applicable to a 2nd order ODE, we need to solve a system of equations in each step as shown below.

$$\begin{aligned}[I - \Delta t A][h_{n+1}] &= h\\ [I - \Delta t A][\dot{h}_{n+1}] &= \dot{h}\end{aligned} \tag{5}$$

2 Runge-Kutta Method

The Runge-Kutta method is fourth order accurate numerical method. In a similar fashion to the Euler method the given Runge-Kutta method can be transformed to be applicable to a 2nd order ODE and the equations are

$$\begin{aligned}h_{n+1} &= h_n + \frac{\Delta t}{6}(k_{1h} + 2k_{2h} + 2k_{3h} + k_{4h})\\ \dot{h}_{n+1} &= \dot{h}_n + \frac{\Delta t}{6}(k_{1\dot{h}} + 2k_{2\dot{h}} + 2k_{3\dot{h}} + k_{4\dot{h}})\end{aligned} \tag{6}$$

where equation 6 combines the K factors in a weighted average calculated at each step.

Test Case

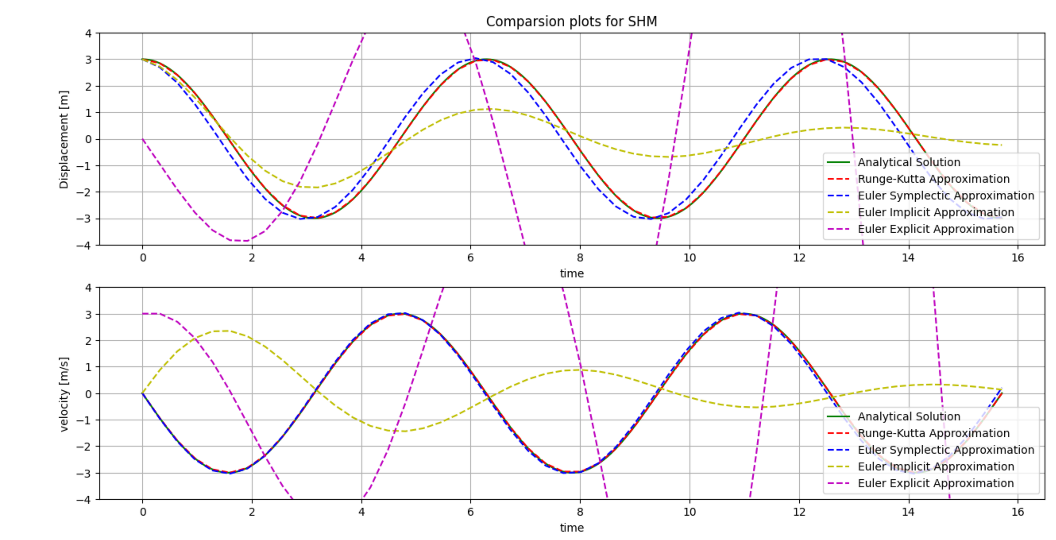

Fig. 2: SHM Test Case.

In Figure 2, it shows the testing of the numerical methods discussed in the previous section. They are used to model the displacement and velocity of a Simple Harmonic Oscillator.

Modeling and Simulation

The following figure shows the modeling of the Euler Symplectic and the Runge-Kutta methods of the booster during the final approach to sea level. It is shown for an ignition time of approximately 36s after the booster was 150km above Earth's surface, traveling at -2km/s, and both methods have a step size of 0.02.

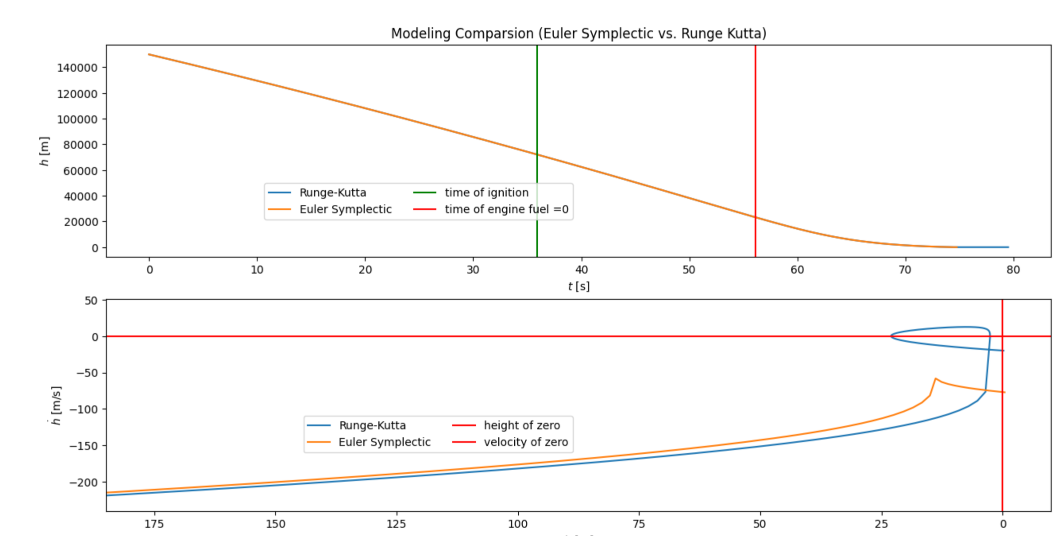

Fig. 3: Booster Motion Modeling

Remarks and Conclusions

The Runge-Kutta method shows a velocity of 0m/s at a height of 3m. These are almost optimum landing parameters. The rocket then has a positive velocity and gains altitude before $m_p = 0$. So, the engine can be throttled down at that time to perform a safe landing. However, when the Euler symplectic model gets close to a height of 0m the rapid changes prove to be too much for the numerical method to adapt with the identical step size. An adaptive step size will be implemented.